A Fictitious domain method
SMARTWING-DYNAMORPH

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Local developments

- CFD
- GETFEM++

- HPC
- Parallelization
Local developments

- CFD
- GETFEM++
- HPC
- Parallelization

- Introduction
  - Laplacian problem
  - Stokes problem
  - Fluid-Structure
  - Numerical results
  - TODO
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CFD
GETFEM++

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Le nouveau système de calcul CALMIP 2009-2013

CALMIP – Altix ICE 8200 EX & UV – 2960 cores – 14 To RAM – 236 To Disks = 33.11 TFlops

Service Fichier Permanent:
Nexis 2000
30 To net
Enhanced NFS sur RF

Service Administration:
1 admin node + console

Service Frontal:
2 Altix 1X 270

Service Visualisation:
4 Altix 5E 500
8 cores, 64 Go RAM
Node Quadro 4000

http://www.calmip.cict.fr/
Three aspects in the project

- theoretical
- experimental
- numerical

The difficulties for numerical part

- complexity of the implementation for coupling model in 2D and 3D
- evolving boundary (interface)
- efficiency (robustness)
- accuracy
- CPU time
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→ Research in Fluid-Structure
Introduction

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Fictitious domain based on Xfem method using Getfem++

⇒ Level-set
⇒ Local assembling
⇒ Optimal error
⇒ Parallel implementation (CALMIP)
The model problem

- $\Omega \in \mathbb{R}^d$ (d=2 or d=3) the computational domain
- $\tilde{\Omega} \in \mathbb{R}^d$ rectangular or parallelepiped domain (the fictitious domain), $\Omega \subset \tilde{\Omega}$
- $\Gamma = \partial \Omega = \Gamma_D \cup \Gamma_N$ ($\Gamma_D$ of nonzero measure, $\Gamma_N$ can be Ø)

Fondamental problem

Find $u : \Omega \rightarrow \mathbb{R}$ such that

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_D \\
\partial_n u &= g \quad \text{on } \Gamma_N
\end{align*}
\]

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ are given

$n$ outward unit vector to $\Gamma$
Weak formulation

- Weak formulation
  \[
  \begin{cases}
  \text{Find } u \in V_0 \text{ such that } \\
  a(u, v) = l(v), \ \forall v \in V_0
  \end{cases}
  \]

  where
  \[
  V = H^1(\Omega), \ V_0 = \{v \in V : v = 0 \text{ on } \Gamma_D\}
  \]

  \[
  a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\Omega \quad \text{and} \quad l(v) = \int_{\Omega} f v d\Omega + \int_{\Gamma_N} g v d\Gamma
  \]

- Equivalent mixed formulation
  \[
  \begin{cases}
  \text{Find } u \in V \text{ and } \lambda \in W \text{ such that } \\
  a(u, v) + \int_{\Gamma_D} \lambda v d\Gamma = l(v), \ \forall v \in V \\
  \int_{\Gamma_D} \mu u d\Gamma = 0, \ \forall \mu \in W
  \end{cases}
  \]

  where \( X = \left\{ w \in L^2(\Gamma_D) : \exists v \in V \text{ such that } w = v|_{\Gamma_D} \right\} \) and \( W = X' \)

- Lagrange multiplier’s interpretation: \( \lambda = -\partial_n u \text{ on } \Gamma_D \)
The mesh

- We define a **regular mesh** including the computational domain.
- The boundary is characterized by **Level-set**.
Fictitious domain approach inspired by Xfem

On the fictitious domain $\tilde{\Omega}$, we consider two finite element spaces $\tilde{V}^h \subset H^1(\tilde{\Omega})$ and $\tilde{W}^h \subset L^2(\tilde{\Omega})$.

Then we define

\[ V^h := \tilde{V}^h|_{\Omega} \quad \text{and} \quad W^h := \tilde{W}^h|_{\Gamma_D} \]

New approximation problem

\[
\begin{aligned}
\text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that }
\end{aligned}
\]

\[
\begin{aligned}
a(u^h, v^h) + \int_{\Gamma_D} \lambda^h v^h d\Gamma &= l(v^h), \quad \forall v^h \in V^h \\
\int_{\Gamma_D} \lambda^h u^h d\Gamma &= 0, \quad \forall \mu^h \in W^h
\end{aligned}
\]
Convergence analysis [Haslinger & Renard (SIAM 2009)]

- Under technical conditions the solution exists and is unique
- For $V^h = \left\{ v^h \in C(\tilde{\Omega}) : v^h_T \in P_k(T) \ \forall T \in \mathcal{T}^h \right\} (k \geq 1)$
  \[
  \inf_{\mu^h \in W^h} ||\lambda - \mu^h||_W \leq h^\beta, \ \beta \geq \frac{1}{2}
  \]
- If $\Gamma = \Gamma_D$ and $u \in H^{1+\frac{d}{2}+\varepsilon}(\Omega) \cap H^1_0(\Omega), \ \varepsilon > 0$
  \[
  ||u - u^h||_V \leq C h^{\frac{1}{2}}, \ h \to 0
  \]

\[1. \ 1|_{\Gamma_D} \in W^h \text{ and } \bar{\mu}^h \in W^h : \int_{\Gamma_D} \bar{\mu}^h v^h d\Gamma = 0, \ \forall v^h \in V^h \implies \bar{\mu}^h = 0\]
**A stabilized problem** [Barbosa & Hugues (CMAME 1991)]

- **Initial problem** (Find a saddle point of the Lagrangian on \( V \times W \))

  \[
  \mathcal{L}(v, \mu) = \frac{1}{2} a(v, v) + \int_{\Gamma_D} \mu v d\Gamma_D - l(v)
  \]

- **Stabilized problem**

  \[
  \mathcal{L}_h(v^h, \mu^h) = \mathcal{L}(v^h, \mu^h) - \frac{\gamma}{2} \int_{\Gamma_D} (\mu^h + \partial_n v^h)^2 d\Gamma, \quad v^h \in V^h, \mu^h \in W^h
  \]

  where \( \gamma := h \gamma_0 \)
A stabilized formulation

Stabilized discrete problem

\[
\begin{aligned}
\text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that } \\
& a(u^h, v^h) + \int_{\Gamma_D} \lambda^h v^h d\Gamma - \gamma \int_{\Gamma_D} \left( \lambda^h + \partial_n u^h \right) \partial_n v^h d\Gamma = l(v^h), \quad v^h \in V^h \\
& \int_{\Gamma_D} \mu^h u^h d\Gamma - \gamma \int_{\Gamma_D} \left( \lambda^h + \partial_n u^h \right) \mu^h d\Gamma = 0, \quad \forall \mu^h \in W^h
\end{aligned}
\]

We define \( B_h : (V^h \times W^h)^2 \rightarrow \mathbb{R} \) by

\[
B_h(u^h, \lambda^h; v^h, \mu^h) := a(u^h, v^h) + \int_{\Gamma_D} \lambda^h v^h d\Gamma + \int_{\Gamma_D} \mu^h u^h d\Gamma \]

\[-\gamma \int_{\Gamma_D} \left( \lambda^h + \partial_n u^h \right) \left( \mu^h + \partial_n v^h \right) d\Gamma\]

then the equivalent discret problem

\[
\begin{aligned}
\text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that } \\
& B_h(u^h, \lambda^h; v^h, \mu^h) = l(v^h), \quad \forall (v^h, \mu^h) \in V^h \times W^h
\end{aligned}
\]
**Convergence analysis** [Haslinger & Renard (SIAM 2009)]

- Under the same assumptions and $\gamma_0$ sufficiently small an Inf-Sup condition is satisfied that ensures existence and uniqueness of the solution (using the norm $|||(z^h, \eta^h)|||^2 := ||z^h||^2 + h^{-1}||z^h||^2_{0, \Gamma_D} + h||\eta^h||^2_{0, \Gamma_D}$)

- For Finite Element Method

  $$\tilde{V}^h = \{ v^h \in C(\bar{\Omega}) : v^h_{|T} \in P_{k_u}(T) \ \forall \ T \in T^h \}, \ k_u \geq 1$$

  $$\tilde{W}^h = \{ \mu^h \in L^2(\bar{\Omega}) : \mu^h_{|T} \in P_{k_\lambda}(T) \ \forall \ T \in T^h \}, \ k_\lambda \geq 0$$

Under an assumption on the intersection of the mesh with $\Omega$, for a solution $u \in H^{k+1}(\Omega)$ and $\lambda \in H^{k-\frac{1}{2}}(\Gamma_D)$ where $k = min\{k_u, k_\lambda + 1\}$

$$|||(u - u^h, \lambda - \lambda^h)||| \leq C h^k ||u||_{k+1, \Omega}$$

---

2. assumption required $h^{\frac{1}{2}} \|\partial_n v^h\|_{0, \Gamma_D} \leq C \|\nabla v^h\|_{0, \Omega}, \ \forall v^h \in V^h, \ \forall h > 0$
The intersection of the mesh with $\Omega$

- We suppose there exists a radius $\hat{\rho} > 0$ independent of $h$ such that $\forall T \in T^h, \ T \cap \Omega \neq \emptyset$ the reference element $\hat{T}$ contains a ball $B(\hat{y}_T, \hat{\rho})$ which satisfies $B(\hat{y}_T, \hat{\rho}) \subset \tau_T^{-1}(T \cap \Omega)$ where $\tau$ is a regular affine transformation in $\mathbb{R}^d$.
- In the figure $T$ is a "bad" element because its intersection with $\Omega$ is small.

smaller is the thickness of the intersection, poorer approximation of the normal derivative on $T \cap \partial \Omega$ is obtained using $v^h_T$

- We choose a neighbor element $T'$ and evaluate the normal derivative from a natural extension of $v^h$ from $T'$ on $T$ then no assumption required
Stokes problem

- Stokes problem (possible extension to Navier-Stokes)

\[
\begin{cases}
-\nu \Delta u + \nabla p &= f \quad \text{in } \mathcal{F} \\
\text{div}(u) &= 0 \quad \text{in } \mathcal{F} \\
u \Delta u + \nabla p &= 0 \quad \text{on } \partial\mathcal{O} \\
u \Delta u + \nabla p &= g \quad \text{on } \partial\mathcal{S}
\end{cases}
\]

- Augmented Lagrangian

\[
\mathcal{L}(u, p, \lambda) = \mathcal{L}_0(u, p, \lambda) - \frac{\gamma}{2} \int_{\partial\mathcal{S}} |\lambda - \sigma(u, p)n|^2 \, d\Gamma
\]

where

\[
\mathcal{L}_0(u, p, \lambda) = \nu \int_{\mathcal{F}} |D(u)|^2 \, d\mathcal{F} - \int_{\mathcal{F}} p\text{div}(u) \, d\mathcal{F} - \int_{\mathcal{F}} f \cdot u \, d\mathcal{F} - \int_{\partial\mathcal{S}} \lambda \cdot (u - g) \, d\Gamma
\]

\[
\sigma(u, p)n = 2\nu D(u)n - pn \quad \text{and} \quad D(u) = \frac{1}{2} \left( \nabla u + \nabla u^T \right)
\]
Fictitious domain

The stabilized formulation of the problem is

\[
\begin{cases}
\text{Find } (u, p, \lambda) \in V \times L^2_0(F) \times H^{1/2}(\partial S) \text{ such that } \\
A((u, p, \lambda); v) = L(v) \quad \forall v \in V_0, \\
B((u, p, \lambda); q) = 0 \quad \forall q \in L^2_0(F), \\
C((u, p, \lambda); \mu) = g(\mu), \quad \forall \mu \in H^{-1/2}(\partial S),
\end{cases}
\]

where

\[
A((u, p, \lambda); v) = 2\nu \int_F D(u) : D(v) dF - \int_F \nu \text{div}(v) dF - \int_{\partial S} \lambda \cdot v d\Gamma \\
- 4\nu^2 \gamma \int_{\partial S} (D(u)n) \cdot (D(v)n) d\Gamma + 2\nu \gamma \int_{\partial S} p(D(v)n \cdot n) d\Gamma + 2\nu \gamma \int_{\partial S} \lambda \cdot (D(v)n) d\Gamma,
\]

\[
B((u, p, \lambda); q) = -\int_F q \text{div}(u) dF + 2\nu \gamma \int_{\partial S} q(D(u)n \cdot n) d\Gamma - \gamma \int_{\partial S} p q d\Gamma - \gamma \int_{\partial S} q \lambda \cdot n d\Gamma,
\]

\[
C((u, p, \lambda); \mu) = -\int_{\partial S} \mu \cdot u d\Gamma + 2\nu \gamma \int_{\partial S} \mu \cdot (D(u)n) d\Gamma - \gamma \int_{\partial S} p(\mu \cdot n) d\Gamma - \gamma \int_{\partial S} \lambda \cdot \mu d\Gamma.
\]

the extension from Laplacian to Stokes and Navier-Stokes is in progress
Fluid-Structure interaction

- Moving solid occupying a time-depending domain $s(t)$
- The displacement of a rigid solid is given by

$$X(y,t) = h(t) + R(t)y, \quad y \in s(0),$$

$$s(t) = h(t) + R(t)s(0),$$

where $h(t)$ is the gravity center and $R(t)$ the rotation

$$
\begin{pmatrix}
\cos(\theta(t)) & -\sin(\theta(t)) \\
\sin(\theta(t)) & \cos(\theta(t))
\end{pmatrix}
$$

- Find $u$, $p$, $h(t)$ and its angular velocity $\theta'(t) = \omega(t)$ (a scalar in 2D)
- Coupling at the interface $\partial s$ using the Dirichlet condition

$$u(x,t) = h'(t) + \theta'(t)(x - h(t)), \quad x \in \partial s(t),$$

with

$$Mh''(t) = -\int_{\partial s(t)} \sigma(u,p)n d\Gamma$$

$$l\theta''(t) = -\int_{\partial s(t)} (x - h(t)) \wedge \sigma(u,p)n d\Gamma$$

accurate computation of $\sigma(u,p)n$ is crucial
Numerical results

- Regular mesh

Finite Element Methods
- P1+/P1/P1
- P2/P1/P1
- Q1/Q0/Q0
- Q2/Q1/Q0

- $\Omega = [0, 1] \times [0, 1]$ and $\partial S$ the circle $(x - 0.5)^2 + (y - 0.5)^2 = R^2$ with $R = 0.21$

- The exact solutions for Stokes problem

  $u_{ex}(x, y) = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{pmatrix}$

  $p_{ex}(x, y) = (y - 0.5) \cos(2\pi x) + (x - 0.5) \sin(2\pi y)$
Rates of convergence without stabilization $u$

$$\| u - u^h \|_{L^2(F)}$$
Rates of convergence without stabilization $p$

\[ \| p - p^h \|_{L^2(\mathcal{F})} \]
Rates of convergence without stabilization $\lambda$

\[ \| \lambda - \lambda^h \|_{L^2(\partial S)} \]
Rates of convergence with stabilization $u$

\[ \left\| u - u^h \right\|_{L^2(\mathcal{F})} \]
Rates of convergence with stabilization $\rho$

\[ \| \rho - \rho^h \|_{L^2(\mathcal{F})} \]
Rates of convergence with stabilization $\lambda$

\[ \| \lambda - \lambda^h \|_{L^2(\partial S)} \]
Numerical tests for Fluid-Structure
Getfem++ is a generic C++ Finite Element library that allows us to consider an independent implementation from the dimension of the problem 2D or 3D (at least for classical model)

⇒ validations of the generic programation "must" be done
⇒ tests "must" be done to study the memory required

The development of the Getfem++ library for CFD, on HPC is in perpetual improvement (parallelization aspect is developed on Calmip)

⇒ parallel tests "must" be done to verify the full parallelization

Efficiency of the fictitious method: to approximate evolving boundary (fluid-structure) the assembling "must" be actualized locally near the interface

⇒ post-doc (time programation) is reserved in the project
State of our work

- Implementation and validation of the stationary Stokes problem
  - fictitious domain without stabilization: YES
  - fictitious domain with stabilization: YES
  - fictitious domain with stabilization and choice of optimal triangle: YES
- Implementation of the time dependent Stokes problem: YES
  - fictitious domain without stabilization: IN PROGRESS
  - fictitious domain with stabilization: IN PROGRESS
  - fictitious domain with stabilization and choice of optimal triangle: IN PROGRESS
- Fluid-Structure (using Stokes model): IN PROGRESS
- Navier-Stokes: NO
- Fluid-Structure (using Navier-Stokes model): NO
- The coupling (structure model): NO

Thank you